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Remarks on E. A. Rahmanov's Paper "On the Asymptotics of the Ratio of Orthogonal Polynomials"*

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It is pointed out that the proof of the basic result of Rahmanov's paper has a serious gap. It is documented by original sources that a statement he relied on in the proof contains a misprint, and it is shown by a counterexample that this statement (with the misprint) is, in fact, false. A somewhat weaker statement is proved true.

1. INTRODUCTION

While working on developing the theory of generalized Pollaczek polynomials, the second author noticed that Theorem 2 of E. A. Rahmanov's paper ([6, p. 247], [7, p. 208] in English translation) would be very helpful in his investigations. Unfortunately, it turned out that the proof of this theorem contained a significant, well-hidden error. We do not claim that Rahmanov's result is false. In fact, we hope it is true; but at this point we must consider it unproven. Owing to the potential significance of this result, as attested by several papers quoting it (cf., e.g., [5, 8, 9, and 10]), we feel it is our duty to draw attention to the error found in [6] and [7].

Rahmanov starts his paper [6, p. 237] (quoted from the English translation [7, p. 199]) as follows:

"Let $\rho(x)$ be a nondecreasing function on the interval $[-1, 1]$ with an

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infinite set of growth points, and let $P_n(x) = x^n + \cdots$ $(n = 0, 1, 2, \ldots)$ be the corresponding orthogonal polynomials

$$\int_{-1}^{1} P_n(x)\, x^\nu \, d\rho(x) = 0, \qquad \nu = 0, 1, \ldots, n-1.$$

In what follows, $D = C \setminus [-1, 1]$ and $\psi(z) = z + \sqrt{z^2 - 1}$, $z \in D$, where the branch of the square root is chosen so that $|\psi(z)| > 1$, $z \in D$.

"The analysis of a series of questions connected with orthogonal polynomials can be substantially simplified in the presence of asymptotics of the form

$$\frac{P_{n+1}(z)}{P_n(z)} \rightrightarrows \frac{1}{2}\, \psi(z), \qquad z \in D. \tag{1}$$

Here and in what follows, $f_n(z) \rightrightarrows f(z)$, $z \in D$, denotes that the sequence $\{f_n\}_1^\infty$ is uniformly convergent to the function f inside (on compact subsets of) the region D."

Then he writes ([6, p. 237, line 2 from below] and [7, p. 200, line 2]):

"The basic result of this paper (Theorem 2) is that (1) also holds in the case when $\rho'(x) > 0$ almost everywhere on $[-1, 1]$."

In fact, Theorem 2 given on page 247 of [6] (p. 208 of [7]) reads as follows:

"THEOREM 2. *Suppose that $\rho' > 0$ almost everywhere on $[-1, 1]$. Then, for the corresponding sequence of orthogonal polynomials,*

$$\frac{P_{n+1}(z)}{P_n(z)} \rightrightarrows \frac{1}{2}\, \psi(z), \qquad z \in D."$$

In attempting to prove Theorem 2, Rahmanov proceeds as follows. First he proves a similar assertion for orthogonal polynomials on the unit circle. Then, using a "well-known" result, he exploits the close relationship between orthogonal polynomials on the real line and those on the unit circle. This result would indeed imply Theorem 2. This is where the error is committed, as this result is false. It is the following statement:

Let $\Phi_n(z)$ $(n = 0, 1, 2, \ldots,)$ be the orthogonal polynomials with leading coefficient 1 on the unit circle with respect to a positive finite Borel measure $d\mu$ on the unit circle that is not confined to finitely many atoms. Then

$$\lim_{n \to \infty} \Phi_{n+1}(z)/\Phi_n(z) = z \text{ uniformly on all compact subsets of the region}$$
$$\{z : |z| > 1\} \text{ if and only if } \lim_{n \to \infty} \Phi_n(0) = 0 \tag{2}$$

(see [6, p. 246, lines 11–8 from below] and [7, p. 207, lines 12–9 from below]). In [6], the reference for (2) is given as [12, p. 376], while in the English translation [7] the reference becomes [11, Sect. 16.4]. However, a close examination of page 376 of [12] reveals that the reference to this page is a misprint. The intended reference is page 467 of [12], where formula (XII.10) is indeed the same as (2). This formula does not appear in [11], the English original of [12], as page 467 of [12] is part of an appendix added only in the Russian translation. An English translation of this appendix appears as [4], and formula (XII.10) occurs on page 96. Unfortunately, no proof of (XII.10) is given in [4] or [12], nor is any reference mentioned. Beyond reasonable doubt, the original source of this formula appears to be Table I, No. 2, in [1, p. 124] ([2, p. 4] and [3, p. 81] in English translation). However, the result stated there says that

$$\lim_{n\to\infty} \Phi_{n+1}(z)/(z\Phi_n(z)) = 1 \text{ uniformly for } |z| \geqslant 1 \quad \text{if and only if } \lim_{n\to\infty} \Phi_n(0) = 0. \tag{3}$$

That is, (2) contains the unfortunate misprint of substituting $>$ for $\geqslant$. As we shall see below, (3) is indeed true, while (2) is false. Thus, Rahmanov must prove that

$$\lim_{n\to\infty} \Phi_{n+1}(z)/(z\Phi_n(z)) = 1 \text{ uniformly for } |z| \geqslant 1 \tag{4}$$

if he wants to conclude that

$$\lim_{n\to\infty} \Phi_n(0) = 0.$$

However, he does not prove (4) in [6] or [7]. Instead, he only proves the following:

"THEOREM 1. *If* $\mu' > 0$ *almost everywhere on* $[0, 2\pi]$, *then*

$$\frac{\Phi_{n+1}(z)}{\Phi_n(z)} \rightrightarrows z, \qquad |z| > 1."$$

(See [6, p. 244] and [7, p. 205]; here μ is a nondecreasing function on the interval $[0, 2\pi]$, which is thought of as the circumference of the unit circle, and the measure $d\mu$ is the associated Lebesgue–Stieltjes measure).

Conclusion. Rahmanov's proof of the basic result (Theorem 2) in [6, p. 247] and [7, p. 208] is not acceptable.

2. The True Statement

It is easy to see that (3) is true. A proof is given in Section 2 of [1], starting with formula (2.8) ([2, 3] in English translation). For the convenience of the reader, we include the proof here. We start with a few remarks that will be useful later as well. As is well known, the polynomials Φ_n satisfy the recurrence formula

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{a}_n \Phi_n^*(z), \qquad \Phi_0(z) = 1, (n \geqslant 0), \tag{5}$$

where the bar indicates complex conjugate, and

$$\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})} \qquad (n \geqslant 0) \tag{6}$$

(cf. [4, (XI.2) on p. 90]). The numbers a_n depend on the measure $d\mu$ with respect to which these polynomials are orthogonal, and, as is clear from (5), we have

$$a_n = -\overline{\Phi_{n+1}(0)} \qquad (n \geqslant 0). \tag{7}$$

It can be shown that

$$|a_n| < 1 \qquad (n \geqslant 0), \tag{8}$$

and otherwise the numbers a_n can be arbitrary. That is, for any choice satisfying (8) of the complex numbers a_n, there is a positive finite Borel measure $d\mu$ on the unit circle that is not confined to finitely many atoms such that the numbers a_n can be obtained via (7) from the polynomials Φ_n orthogonal with respect to $d\mu$ (cf. (XI.9) and (XI.10) and the text in between on pp. 91–92 in [4]). It follows by induction on n from (5) and (8) and Rouché's Theorem that

$$\text{all roots of } \Phi_n \text{ are inside the unit disk} \quad (n \geqslant 0). \tag{9}$$

Proof of (3). To show the "if" part of (3), assume that the right-hand side of the biconditional in (3) is true, which means, according to (7), that

$$\lim_{n \to \infty} a_n = 0. \tag{10}$$

As $\Phi_n^*(z)/(z\Phi_n(z))$ is holomorphic for $|z| \geqslant 1$ (including the point $z = \infty$; cf. (9)) and is equal to 1 for $|z| = 1$ (cf. (6)), it follows from the Maximum Principle that this function has absolute value $\leqslant 1$ for $|z| \geqslant 1$. Hence (5) implies that

$$|\Phi_{n+1}(z)/(z\Phi_n(z)) - 1| \leqslant |a_n|.$$

Thus the left-hand side of the biconditional in (3) follows from (10).

To establish the "only if" part, note that (5) with $z=1$ implies

$$\Phi_{n+1}(1)/\Phi_n(1)=1-\bar{a}_n\Phi_n^*(1)/\Phi_n(1).$$

If we assume that the left-hand side of the biconditional in (3) is valid, then, substituting $z=1$, it follows that the limit of the left-hand side here is 1. As $|\Phi_n^*(1)/\Phi_n(1)|=1$ (cf. (6)), this entails that $\lim_{n\to\infty} a_n=0$. In view of (7), this establishes the only if part of (3). The proof of (3) is complete.

3. The False Statement

Next we are going to give an example showing that (2) is false. To this end, we will choose the numbers a_n such that for any $\eta>1$ we will have

$$\lim_{n\to\infty} \Phi_{n+1}(z)/(z\Phi_n(z))=1 \text{ uniformly for } |z|\geqslant\eta, \tag{10}$$

and yet

$$\lim_{n\to\infty} \Phi_n(0) \text{ does not exist.} \tag{11}$$

In order to do this, we will first construct a fast-increasing sequence of positive integers n_k and require that

$$a_n=0 \qquad \text{unless } n=n_k \text{ for some } k\geqslant 0, \tag{12}$$

and otherwise a_n is arbitrary, subject to the stipulation in (8). We can conclude from (12) by (5) and (6) that for $n_k\leqslant n<n_{k+1}$

$$\Phi_{n+1}(z)=z\Phi_n(z)=z^{n-n_k}\Phi_{n_k+1}(z) \tag{13}$$

and

$$\Phi_{n+1}^*(z)=\Phi_{n_k+1}^*(z)$$

hold. The latter formula in conjunction with (5) implies that

$$\Phi_{n_{k+1}+1}(z)=z\Phi_{n_{k+1}}(z)-\bar{a}_{n_{k+1}}\Phi_{n_k+1}^*(z). \tag{14}$$

The sequence $\langle a_n\rangle$ to be constructed is subject only to stipulations (8) and (12), and so we can choose this sequence such that $\lim_{n\to\infty} a_n$ does not exist, i.e., such that (11) is satisfied (cf. (7)); to this end, we can take, e.g., $a_{n_k}=1/2$. And yet, we are going to show that for a suitable choice of the sequence $\langle n_k \rangle$ (10) will be satisfied; this will show that (2) is indeed false. In view of (13) and (14), for this we will have to show only that, for every $\eta > 1$, we have

$$\lim_{k\to\infty} \Phi^*_{n_k+1}(z)/(z\Phi_{n_{k+1}}(z)) = 0$$

uniformly for $|z| \geqslant \eta$ (notice $|a_{n_k}| < 1$ according to (8)). Rewriting the denominator here by using (13), we see that this is equivalent to saying that, for every $\eta > 1$,

$$\lim_{k\to\infty} \Phi^*_{n_k+1}(z)/(z^{n_{k+1}-n_k}\Phi_{n_k+1}(z)) = 0 \tag{15}$$

uniformly for $|z| \geqslant \eta$.

To establish this for an appropriate choice of the sequence $\langle n_k \rangle$, let η_k be a decreasing sequence of reals with $\eta_k > 1$ and $\lim_{k\to\infty} \eta_k = 1$. Let $|z| \geqslant \eta$ for some $\eta > 1$, and fix k_0 so large that $\eta \geqslant \eta_{k_0}$. Then we have

$$|z| \geqslant \eta_k > 1 \qquad \text{for} \quad k \geqslant k_0. \tag{16}$$

We are going to estimate the numerator and the denominator in (15) for $k \geqslant k_0$. To estimate the numerator, first note that (13) and (14) imply that

$$\Phi_{n_{k+1}+1}(z) = z^{n_{k+1}-n_k}\Phi_{n_k+1}(z) - \bar{a}_{n_{k+1}}\Phi^*_{n_k+1}(z)$$

holds for all $k \geqslant 0$. This (together with (6)) can be considered as a recurrence equation defining the polynomials Φ_{n_k+1}. If we choose the integers n_k such that

$$n_{k+1} \geqslant 2n_k + 2 \tag{17}$$

for all $k \geqslant 0$, then the two terms on the right-hand side contain no common powers of z; that is, the coefficients of powers of z in $\Phi_{n_k+1}(z)$ do not add up. Therefore (8) and equation $\Phi_0(z) = 1$ in (5) imply that all the coefficients in $\Phi_{n_k+1}(z)$ have absolute values $\leqslant 1$. As $\Phi^*_{n_k+1}(z)$ has the same coefficients (in reverse order), we have, for $k \geqslant k_0$, that

$$|\Phi^*_{n_k+1}(z)| \leqslant \sum_{j=0}^{n_k+1} |z^j| \leqslant z^{n_k+2}/(\eta_k - 1), \tag{18}$$

where the second inequality holds by virtue of (16). To estimate the denominator in (15) for $k \geqslant k_0$, note that the leading coefficient of $\Phi_{n_k+1}(z)$ is 1; hence we have the factorization

$$\Phi_{n_k+1}(z) = \prod_{j=1}^{n_k+1} (z - \zeta_j),$$

where the roots ζ_j of $\Phi_{n_k+1}(z)$ satisfy $|\zeta_j| < 1$ according to (9). In view of (16), this means that

$$|\Phi_{n_k+1}(z)| \geqslant (\eta_k - 1)^{n_k+1} \qquad (k \geqslant k_0).$$

This, together with (18), implies that the expression after the limit in (15) can be estimated for $k \geqslant k_0$ as

$$\begin{aligned} |\Phi^*_{n_k+1}(z)/(z^{n_{k+1}-n_k}\Phi_{n_k+1}(z))| &\leqslant |z|^{2+2n_k-n_{k+1}}/(\eta_k - 1)^{n_k+2} \\ &\leqslant \eta_k^{2+2n_k-n_{k+1}}/(\eta_k - 1)^{n_k+2}, \end{aligned} \tag{19}$$

where the second inequality can be seen to hold by (16) and (17) (the latter is needed to ensure that the exponent of η_k on the right-hand side is not positive. Given n_k, choose n_{k+1} such that the right-hand side here is less than, say, $1/k$. That is, choose the positive integers n_k such that

$$n_{k+1} \geqslant 2n_k + 2$$

and

$$\eta_k^{2+2n_k-n_{k+1}}/(\eta_k - 1)^{n_k+2} < 1/k \qquad (k \neq 0)$$

hold for all $k \geqslant 0$ (the first formula here is identical to (17)). Then, according to (19), the expression after the limit in (15) will be $< 1/k$ provided k is large enough for (16) to be satisfied, i.e., if $\eta_k \leqslant \eta$. This shows that (15) holds uniformly for $|z| \geqslant \eta$. This completes the proof that (2) is false.

4. A Weaker Statement

As we just showed, the "only if" part of (2) is false. Here we establish a weaker implication.

THEOREM. *Suppose that*

$$\lim_{n\to\infty} \Phi_{n+1}(z)/\Phi_n(z) = z \tag{20}$$

holds for all z *with* $|z| > 1$. *Then*

$$\lim_{n\to\infty} \Phi_n(0) \cdot \Phi_{n+1}(0) = 0. \tag{21}$$

It is not difficult to construct an example showing that the converse is not true.

Proof. Noting that $\Phi_n^*(z)$ has no zeros in the unit disk according to (9) and (6), it follows from the Theorem of Residues that

$$\frac{1}{2\pi i}\oint_{|z|=1/2} \frac{a_n \Phi_n(z)}{z\Phi_n^*(z)}\,dz = a_n \Phi_n(0)/\Phi_n^*(0)$$
$$= -\Phi_n(0)\,\overline{\Phi_{n+1}(0)} \tag{22}$$

according to (7) and the observation that $\Phi_n^*(0)$ equals the leading coefficient of $\Phi_n(z)$, which in turn equals 1 (cf. (5)). Notice also that the absolute value of the integrand here is $\leqslant 2$. Indeed, $|a_n| < 1$ according to (8), and

$$|\Phi_n(z)/\Phi_n^*(z)| \leqslant 1 \qquad \text{for} \quad |z| = 1/2$$

by virtue of the Maximum Principle, since $|\Phi_n(z)/\Phi_n^*(z)| = 1$ for $|z| = 1$ in view of (6). By means of the formula

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - a_n z \Phi_n(z),$$

which is an easy consequence of (5) and (6), we can conclude that the integrand in (22) equals

$$z^{-2}(1 - \Phi_{n+1}^*(z)/\Phi_n^*(z)),$$

and, for $|z| = 1/2$, say, this has limit 0 as $n \to \infty$ in view of (20) (and (6)). Hence (21) follows from (22) via Lebesgue's Bounded Convergence Theorem, completing the proof.

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